CCFU Proof 7

Arctangent Identity: $\arctan(1/\varphi) + \arctan(1/\varphi^3) = \pi/4$

Given. Let $\varphi = (1+\sqrt{5})/2$. Then $\varphi^2 = \varphi + 1$ and $\varphi > 1$.

Claim.

$$\arctan\left(\frac{1}{\varphi}\right) + \arctan\left(\frac{1}{\varphi^3}\right) = \frac{\pi}{4}.$$

Proof. Using the addition formula

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right) \quad \text{when } ab < 1.$$

Let $a = 1/\varphi$, $b = 1/\varphi^3$.

Branch safety. $ab = 1/\varphi^4 < 1$. Since $a > 0$, $b > 0$, and $ab < 1$, the sum lies in $(0, \pi/2)$, so no branch correction is needed.

Numerator:

$$a + b = \frac{1}{\varphi} + \frac{1}{\varphi^3} = \frac{\varphi^2+1}{\varphi^3} = \frac{\varphi+2}{\varphi^3}.$$

Denominator:

$$1 - ab = 1 - \frac{1}{\varphi^4} = \frac{\varphi^4 - 1}{\varphi^4}.$$

Powers of φ. Using $\varphi^2 = \varphi + 1$:

$$\varphi^3 = 2\varphi + 1, \qquad \varphi^4 = 3\varphi + 2, \qquad \varphi^4 - 1 = 3\varphi + 1.$$

Ratio:

$$\begin{aligned}\frac{a+b}{1-ab} &= \frac{(\varphi+2)/\varphi^3}{(3\varphi+1)/\varphi^4} = \frac{\varphi(\varphi+2)}{3\varphi+1} \\ &= \frac{\varphi^2+2\varphi}{3\varphi+1} = \frac{(\varphi+1)+2\varphi}{3\varphi+1} = \frac{3\varphi+1}{3\varphi+1} = 1.\end{aligned}$$

Therefore:

$$\arctan\left(\frac{1}{\varphi}\right) + \arctan\left(\frac{1}{\varphi^3}\right) = \arctan(1) = \frac{\pi}{4}. \qquad \blacksquare$$